

# Fisher information distance: a geometrical reading\*

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October 9, 2012

## Abstract

This paper is a strongly geometrical approach to the Fisher distance, which is a measure of dissimilarity between two probability distribution functions. This, as well as other divergence measures, are also used in many applications to establish a proper data average. It focuses on statistical models of the normal probability distribution functions and takes advantage of the connection with the classical hyperbolic geometry to derive closed forms for the Fisher distance in several cases. Connections with the well-known Kullback-Leibler divergence measure are also devised. The main purpose is to widen the range of possible interpretations and relations of the Fisher distance and its associated geometry for the prospective applications, in particular to information theory.

**Keywords:** Fisher distance, information geometry, normal probability distribution functions, Kullback-Leibler divergence, hyperbolic geometry.

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\*Partially supported by FAPESP (Grants 2007/56052-8, 2007/00514-3 and 2011/01096-6), CNPq (Grants 309561/2009-4 and 304032/2010-7) and PRONEX-Optimization.

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# 1 Introduction

Information geometry is a research field that has provided framework and enlarged the perspective of analysis for a wide variety of domains, such as statistical inference, information theory, mathematical programming, neurocomputing, to name a few. It is an outcome of the investigation of the differential geometric structure on manifolds of probability distributions, with the Riemannian metric defined by the Fisher information matrix [1]. Rao's pioneering work [16] was subsequently followed by several authors (e.g. [2, 17, 12], among others). We quote [1] as a general reference for this matter.

Concerning specifically to information theory and signal processing, an important aspect of the Fisher matrix arises from its trace being related to the surface area of the typical set associated with a given probability distribution, whereas the volume of this set is related to the entropy. This was used to establish connections between inequalities in information theory and geometric inequalities ([5], [8]).

In general, many applications demand a measure of dissimilarity between the distributions of the involved objects, or also require the replacement of a set of data by a proper average or a centroid. In both cases, the Fisher distance may apply as well as other dissimilarity measures ([11, 14, 15]).

Our contribution in this paper is to present a geometrical view of the Fisher matrix, focusing on the parameters that describe the univariate and the multivariate normal distributions, with the aim of widen the range of possible interpretations for the prospective applications of information geometry to information theory and other related fields.

Our geometrical reading allowed to employ results from the classical hyperbolic geometry and to derive closed expressions for the Fisher distance in special cases of the multivariate normal distributions. A preliminary version of some results presented here has appeared in [6].

This text is organized as follows: in Section 2 we explore the two dimensional statistical model of the Gaussian (normal) univariate probability distribution function (PDF). Closed forms for this distance are derived in the most common parameters and a relationship with the Kullback-Leibler measure of divergence is presented. Section 3 is devoted to the Fisher information geometry of the multivariate normal PDF's. For the special cases of the round Gaussian distributions and normal distributions with diagonal covariance matrices, closed forms for the distances are derived. We discuss the Fisher information matrix for the general bivariate case as well.

## 2 The hyperbolic model of the mean $\times$ standard deviation half-plane

The geometric model of the mean  $\times$  standard deviation half-plane associates each point in the half upper plane of  $\mathbb{R}^2$  with a univariate Gaussian probability distribution function

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-|x - \mu|^2}{2\sigma^2}\right).$$

Hence, a classic parametric space for this family of PDF's is

$$H = \{(\mu, \sigma) \in \mathbb{R}^2 \mid \sigma > 0\}.$$

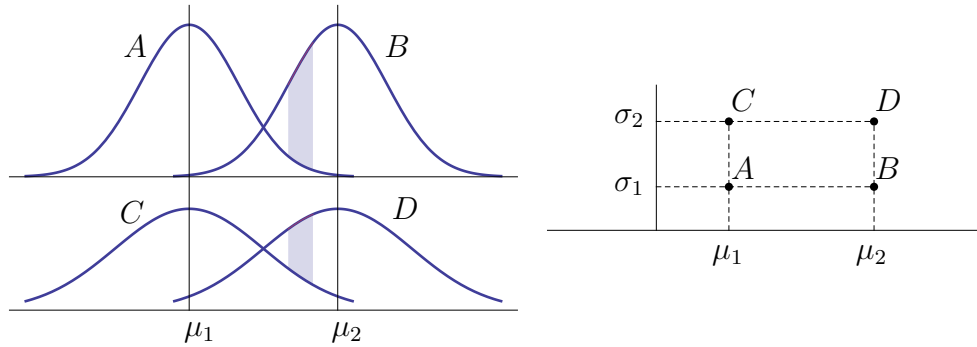


Figure 1: Univariate normal distributions and their representations in the  $(\mu, \sigma)$  half-plane.

A distance between two points  $P = (\mu_1, \sigma_1)$  and  $Q = (\mu_2, \sigma_2)$  in the half-plane  $H$  should reflect the dissimilarity between the associated PDF's. We will not distinguish the notation of the point  $P$  in the parameter space and its associated PDF  $f(x, P)$ .

A comparison between univariate normal distributions is illustrated in Figure 1. By fixing the means and increasing the standard deviation, we can see that the dissimilarity between the probabilities attached to the same interval concerning the PDF's associated with  $C$  and  $D$  is smaller than the one between the PDF's associated with  $A$  and  $B$  (left). This means that the distance between points in the upper half-plane (right) representing normal distributions cannot be Euclidean. Moreover, we can observe that such a metric must vary with the inverse of the standard deviation  $\sigma$ . The points  $C$  and  $D$  should be closer to each other than the points  $A$  and  $B$ , reflecting that the pair of distributions  $A$  and  $B$  is more dissimilar than the pair  $C$  and  $D$ .

A proper distance arises from the Fisher information matrix, which is a measure of the amount of information of the location parameter ([7], ch. 12). For univariate distributions parametrized by an  $n$ -dimensional space, the coefficients of this matrix, which define a metric, are calculated as the expectation of a product involving partial derivatives of the logarithm of the PDF's:

$$g_{ij}(\boldsymbol{\beta}) = \int_{-\infty}^{\infty} f(x, \boldsymbol{\beta}) \frac{\partial \ln f(x, \boldsymbol{\beta})}{\partial \beta_i} \frac{\partial \ln f(x, \boldsymbol{\beta})}{\partial \beta_j} dx.$$

A metric matrix  $G = (g_{ij})$  defines an inner product as follows:

$$\langle u, v \rangle_G = u^T (g_{ij}) v \quad \text{and} \quad \|u\|_G = \sqrt{\langle u, u \rangle_G}.$$

The distance between two points  $P, Q$  is given by the number which is the minimum of the lengths of all the piecewise smooth paths  $\gamma_P^Q$  joining these two points. The length of a path  $\gamma(t)$  is calculated by using the inner product  $\langle \cdot, \cdot \rangle_G$ :

$$\text{Length of } \gamma = \int_{\gamma} ds = \int_{\gamma} \|\gamma'(t)\|_G dt$$

and so

$$d_G(P, Q) = \min_{\gamma_P^Q} \{\text{Length of } \gamma\}.$$

A curve that encompasses this shortest path is a *geodesic*.

In the univariate normally distributed case described above we have  $\boldsymbol{\beta} = (\beta_1, \beta_2) = (\mu, \sigma)$  and it can be easily deduced that the Fisher information matrix is

$$[g_{ij}(\mu, \sigma)] = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix} \quad (1)$$

so that the expression for the metric is

$$ds_F^2 = \frac{d\mu^2 + 2d\sigma^2}{\sigma^2}. \quad (2)$$

The Fisher distance is the one associated with the Fisher information matrix (1). In order to express such a notion of distance and to characterize the geometry in the plane  $\mathbb{H}_F^2$ , we analyze its analogies with the well-known Poincaré half-plane  $\mathbb{H}^2$ , a model for the hyperbolic geometry, the metric of which is given by the matrix

$$[g_{ij}]_H = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix}. \quad (3)$$

The inner product associated with the Fisher matrix (1) will be denoted by  $\langle \cdot, \cdot \rangle_F$  and the distance between  $P = (\mu_1, \sigma_1)$  and  $Q = (\mu_2, \sigma_2)$  in the upper half-plane  $\mathbb{H}_F^2$ , by  $d_F(P, Q)$ . The distance in the Poincaré half-plane induced by (3) will be denoted by  $d_H(P, Q)$ . By considering the similarity mapping  $\Psi : \mathbb{H}_F^2 \rightarrow \mathbb{H}^2$  defined by  $\Psi(\mu, \sigma) = (\mu/\sqrt{2}, \sigma)$ , we can see that

$$d_F((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \sqrt{2} d_H \left( \left( \frac{\mu_1}{\sqrt{2}}, \sigma_1 \right), \left( \frac{\mu_2}{\sqrt{2}}, \sigma_2 \right) \right), \quad (4)$$

Besides, the geodesics in  $\mathbb{H}_F^2$  are the inverse image, by  $\Psi$ , of the geodesics in  $\mathbb{H}^2$ . Vertical half-lines and half-circles centered at  $\sigma = 0$  are the geodesics in  $\mathbb{H}^2$  (see, eg. [3, Ch.7]). Hence, the geodesics in  $\mathbb{H}_F^2$  are half-lines and half-ellipses centered at  $\sigma = 0$ , with eccentricity  $1/\sqrt{2}$ . We can also assert that a circle in the Fisher distance is an ellipse with the same eccentricity and its center is below the Euclidean center. Figure 2 shows the Fisher circle centered at  $A = (1.5, 0.75)$  and radius 2.3769, and the geodesics connecting the center to points  $B$ ,  $E$  and  $F$  on the circle.

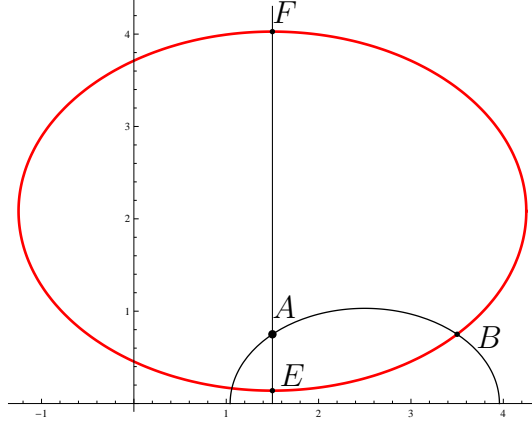


Figure 2: A Fisher circle centered at  $A$  and geodesic arcs  $AB$ ,  $AF$  and  $AE$ , with  $d_F(A, B) = d_F(A, F) = d_F(A, E)$ .

The distance between two points in the Poincaré half-plane can be expressed by the logarithm of the cross-ratio between these two points and the points at the infinite:

$$d_H(P, Q) = \ln(P_\infty, P, Q, Q_\infty).$$

It can be stated by the following formulas, considering  $P$  and  $Q$  as vertical lined or not, as illustrated in Figure 3, respectively:

$$d_H(P, Q) = \ln\left(\frac{\sigma_Q}{\sigma_P}\right) \text{ or } d_H(P, Q) = \ln\left(\frac{PQ_\infty}{PP_\infty} \cdot \frac{QP_\infty}{QQ_\infty}\right) = \ln\left(\frac{\tan\left(\frac{\alpha_P}{2}\right)}{\tan\left(\frac{\alpha_Q}{2}\right)}\right).$$

By recalling that the Fisher distance  $d_F$  and the hyperbolic distance  $d_H$  are related by (4) we obtain the following closed expression for the Fisher information distance:

$$d_F((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \sqrt{2} \ln \frac{\left| \left( \frac{\mu_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\mu_2}{\sqrt{2}}, -\sigma_2 \right) \right| + \left| \left( \frac{\mu_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\mu_2}{\sqrt{2}}, \sigma_2 \right) \right|}{\left| \left( \frac{\mu_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\mu_2}{\sqrt{2}}, -\sigma_2 \right) \right| - \left| \left( \frac{\mu_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\mu_2}{\sqrt{2}}, \sigma_2 \right) \right|} \quad (5)$$

$$= \sqrt{2} \ln \left( \frac{\mathcal{F}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) + (\mu_1 - \mu_2)^2 + 2(\sigma_1^2 + \sigma_2^2)}{4\sigma_1\sigma_2} \right) \quad (6)$$

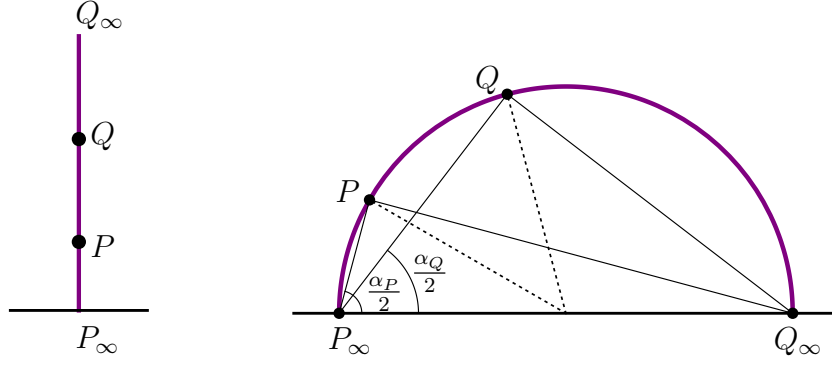


Figure 3: Elements to compute the distance  $d_H(P, Q)$ , in case the points  $P, Q \in \mathbb{H}^2$  are vertically aligned (left) or not (right).

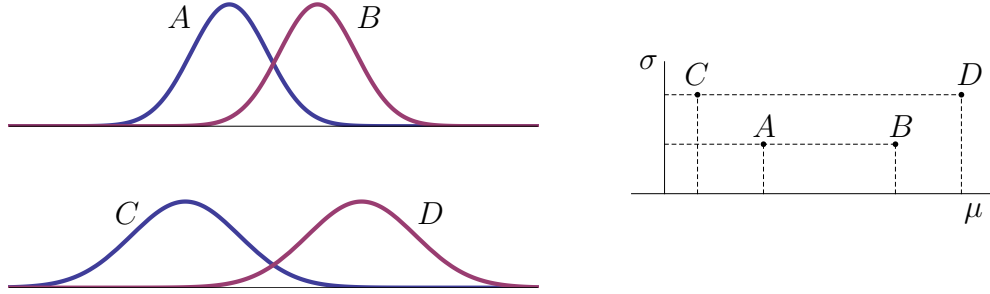


Figure 4: Equidistant pairs in Fisher metric:  $d_H(A, B) = d_H(C, D) = 2.37687$ , where  $A = (1.5, 0.75)$ ,  $B = (3.5, 0.75)$  and  $C = (0.5, 1.5)$ ,  $D = (4.5, 1.5)$ .

where

$$\mathcal{F}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \sqrt{((\mu_1 - \mu_2)^2 + 2(\sigma_1 - \sigma_2))((\mu_1 - \mu_2)^2 + 2(\sigma_1 + \sigma_2))}.$$

Figure 4 illustrates two distinct pairs of Gaussian distributions which are equidistant with the Fisher metric. Moreover, from the relations (5)-(6) we can deduce facts of the geometry of the upper half plane with the Fisher metric: it is hyperbolic with constant curvature equal to  $-\frac{1}{2}$  and the shortest path between the representatives of two normal distributions is either on a vertical line or on a half ellipse (see Figure 6(a)).

The Fisher distance between two PDF's  $P = (\mu, \sigma_1)$  and  $Q = (\mu, \sigma_2)$  is

$$d_F(P, Q) = \sqrt{2} |\ln(\sigma_2/\sigma_1)| \quad (7)$$

and the vertical line connecting  $P$  and  $Q$  is a geodesic in the Fisher half-plane. On the other hand, the geodesic connecting  $P = (\mu_1, \sigma)$  and  $Q = (\mu_2, \sigma)$  associated with two normal PDF's with the same variance is not the horizontal line connecting these points

(the shortest path is contained in a half-ellipse). Indeed,

$$d_F(P, Q) = \sqrt{2} \ln \left( \frac{4\sigma^2 + (\mu_1 - \mu_2)^2 + |\mu_1 - \mu_2| \sqrt{8\sigma^2 + (\mu_1 - \mu_2)^2}}{4\sigma^2} \right) < \frac{|\mu_2 - \mu_1|}{\sigma}. \quad (8)$$

The expression on the right of (8) is the length of the horizontal segment joining  $P$  and  $Q$ . Nevertheless, in case just normal PDF's with constant variance are considered, the expression on the right of (8) is a proper distance.

It is worth mentioning that the Fisher metric can also be used to establish the concept of *average* distribution between two given distributions  $A$  and  $Q$ . This is determined by the point  $M$  on the geodesic segment joining  $A$  and  $Q$  and which is equidistant to these points in Figure 5.

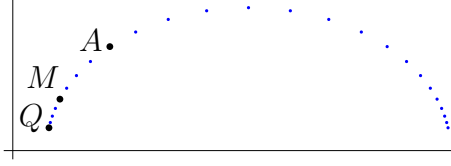


Figure 5: The Fisher average between distributions  $A = (1.5, .75)$  and  $Q = (1.0610, 0.1646)$  is  $M = (1.1400, 0.3711)$ . The plotted points form a polygonal with equal Fisher length segments.

## 2.1 Univariate normal distributions described in other usual parameters

Univariate normal distributions may be also described by means of the so-called *source*  $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}_+$ , *natural*  $(\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}_-$  and *expectation* parameters  $(\eta_1, \eta_2) \in \mathbb{R} \times \mathbb{R}_+$ , respectively defined by

$$\begin{aligned} (\lambda_1, \lambda_2) &= (\mu, \sigma^2) \\ (\theta_1, \theta_2) &= \left( \frac{\mu}{\sigma^2}, \frac{-1}{2\sigma^2} \right) \end{aligned}$$

and

$$(\eta_1, \eta_2) = (\mu, \sigma^2 + \mu^2).$$

Therefore,

$$(\mu, \sigma) = (\lambda_1, \sqrt{\lambda_2}) = \left( \frac{-\theta_1}{2\theta_2}, \frac{1}{\sqrt{-2\theta_2}} \right) = \left( \eta_1, \sqrt{\eta_2 - \eta_1^2} \right)$$

and expressions (5)-(6) may be restated, for the source parameters, as

$$\begin{aligned} d_F((\lambda_{11}, \sqrt{\lambda_{21}}), (\lambda_{12}, \sqrt{\lambda_{22}})) &= d_\lambda((\lambda_{11}, \lambda_{21}), (\lambda_{12}, \lambda_{22})) = \\ &\sqrt{2} \ln \left( -\frac{\sqrt{(\lambda_{11} - \lambda_{12})^2 + 2(\sqrt{\lambda_{21}} - \sqrt{\lambda_{22}})^2} + \sqrt{(\lambda_{11} - \lambda_{12})^2 + 2(\sqrt{\lambda_{21}} + \sqrt{\lambda_{22}})^2}}{\sqrt{(\lambda_{11} - \lambda_{12})^2 + 2(\sqrt{\lambda_{21}} - \sqrt{\lambda_{22}})^2} - \sqrt{(\lambda_{11} - \lambda_{12})^2 + 2(\sqrt{\lambda_{21}} + \sqrt{\lambda_{22}})^2}} \right), \end{aligned}$$

for the natural parameters as

$$d_F \left( \left( \frac{-\theta_{11}}{2\theta_{21}}, \frac{1}{\sqrt{-2\theta_{21}}} \right), \left( \frac{-\theta_{12}}{2\theta_{22}}, \frac{1}{\sqrt{-2\theta_{22}}} \right) \right) = d_\theta((\theta_{11}, \theta_{21}), (\theta_{12}, \theta_{22})) =$$

$$\sqrt{2} \ln \left( - \frac{\sqrt{4 \left( \frac{1}{\sqrt{-\theta_{21}}} - \frac{1}{\sqrt{-\theta_{21}}} \right)^2 + \left( \frac{\theta_{11}}{\theta_{21}} - \frac{\theta_{12}}{\theta_{22}} \right)^2} + \sqrt{4 \left( \frac{1}{\sqrt{-\theta_{22}}} + \frac{1}{\sqrt{-\theta_{21}}} \right)^2 + \left( \frac{\theta_{11}}{\theta_{21}} - \frac{\theta_{12}}{\theta_{22}} \right)^2}}{\sqrt{4 \left( \frac{1}{\sqrt{-\theta_{21}}} - \frac{1}{\sqrt{-\theta_{21}}} \right)^2 + \left( \frac{\theta_{11}}{\theta_{21}} - \frac{\theta_{12}}{\theta_{22}} \right)^2} - \sqrt{4 \left( \frac{1}{\sqrt{-\theta_{22}}} + \frac{1}{\sqrt{-\theta_{21}}} \right)^2 + \left( \frac{\theta_{11}}{\theta_{21}} - \frac{\theta_{12}}{\theta_{22}} \right)^2}} \right)$$

and for the expectation parameters as

$$d_F((\eta_{11}, \sqrt{\eta_{21} - \eta_{11}^2}), (\eta_{12}, \sqrt{\eta_{22} - \eta_{12}^2})) = d_\eta((\eta_{11}, \eta_{21}), (\eta_{12}, \eta_{22})) =$$

$$\sqrt{2} \ln \left( - \frac{\sqrt{(\eta_{11} - \eta_{12})^2 + 2 \left( \sqrt{\eta_{21} - \eta_{11}^2} - \sqrt{\eta_{22} - \eta_{12}^2} \right)^2} + \sqrt{(\eta_{11} - \eta_{12})^2 + 2 \left( \sqrt{\eta_{21} - \eta_{11}^2} + \sqrt{\eta_{22} - \eta_{12}^2} \right)^2}}{\sqrt{(\eta_{11} - \eta_{12})^2 + 2 \left( \sqrt{\eta_{21} - \eta_{11}^2} - \sqrt{\eta_{22} - \eta_{12}^2} \right)^2} - \sqrt{(\eta_{11} - \eta_{12})^2 + 2 \left( \sqrt{\eta_{21} - \eta_{11}^2} + \sqrt{\eta_{22} - \eta_{12}^2} \right)^2}} \right).$$

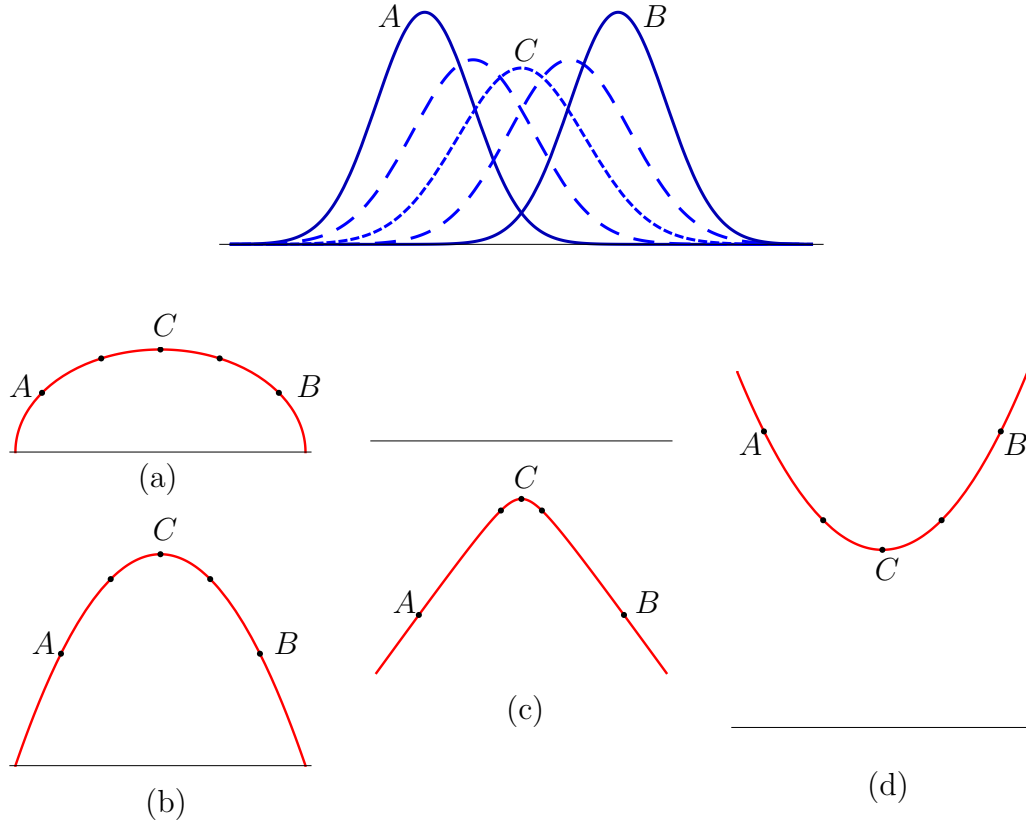


Figure 6: Shortest path between the normal distributions  $A$  and  $B$  in the distinct half-planes: (a) Classic parameters  $(\mu, \sigma)$  – mean  $\times$  standard deviation; (b) Source parameters  $(\mu, \sigma^2)$  – mean  $\times$  variance; (c) Natural parameters  $(\theta_1, \theta_2) = (\frac{\mu}{\sigma^2}, \frac{-1}{2\sigma^2})$  and (d) Expectation parameters  $(\eta_1, \eta_2) = (\mu, \mu^2 + \sigma^2)$ .

The shortest path between two normal distributions is depicted in Figure 6 for the four distinct half-planes, described by the classic (a), the source (b), the natural (c) and



the expectation parameters (d). Besides the half-ellipse that contains the shortest path in the classic mean  $\times$  standard deviation half-plane, the shortest path in the source mean  $\times$  variance and in the expectation half-planes are described by arc of parabolas, whereas an arc of a half-hyperbola contains the shortest path in the natural half-plane.

## 2.2 The Kullback-Leibler divergence and the Fisher distance

Another measure of dissimilarity between two PDF's is the Kullback-Leibler divergence [10], which is used in information theory and commonly referred to as the relative entropy of a probability distribution. It is not a distance neither a symmetric measure. In what follows we discuss its relation with the Fisher distance in the case of univariate normal distributions. Its expression in this case is:

$$KL((\mu_1, \sigma_1) || (\mu_2, \sigma_2)) = \frac{1}{2} \left( 2 \ln \left[ \frac{\sigma_2}{\sigma_1} \right] + \frac{\sigma_1^2}{\sigma_2^2} + \frac{(\mu_1 - \mu_2)^2}{\sigma_2} - 1 \right)$$

A symmetrized version of this measure,

$$\begin{aligned} d_{KL}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) &= \sqrt{KL((\mu_1, \sigma_1) || (\mu_2, \sigma_2)) + KL((\mu_2, \sigma_2) || (\mu_1, \sigma_1))} \\ &= \sqrt{\frac{1}{2} \left( -2 + \frac{(\mu_1 - \mu_2)^2}{\sigma_1} + \frac{\sigma_1^2}{\sigma_2^2} + \frac{(\mu_1 - \mu_2)^2}{\sigma_2} + \frac{\sigma_2^2}{\sigma_1^2} \right)}, \end{aligned}$$

is also used.

If the points in the parameter space are vertically aligned ( $P = (\mu, \sigma_1)$  and  $Q = (\mu, \sigma_2)$ ), the Fisher distance  $d = d_F(P, Q) = \sqrt{2} \ln(\frac{\sigma_2}{\sigma_1})$  from what we get an expression of the Kullback-Leibler divergences in terms of the Fisher distance:

$$KL(P || Q) = g(d) = \frac{1}{2} \left( e^{-\sqrt{2}d} + 2 \ln \left( e^{\frac{d}{\sqrt{2}}} \right) - 1 \right),$$

$$KL(Q || P) = g(-d) \quad \text{and} \quad d_{KL}(P, Q) = \sqrt{\frac{e^{\sqrt{2}d} + e^{-\sqrt{2}d}}{2} - 1} = \sqrt{\cosh(\sqrt{2}d) - 1}.$$

Figure 7 (left) shows the graphics of the mappings  $g(d) = KL(A || Y)$  (red continuous curve),  $g(-d) = KL(Y || A)$  (blue dashed curve), and the symmetrized  $d_{KL}(A, Y)$  (green dot-dashed curve) when  $Y$  goes from  $A$  to  $F$  in Figure 2, compared to the Fisher distance  $d$  (identity), which varies in the interval  $[0, 2.3769]$ .

It is straightforward in this case to prove that the symmetrized Kullback-Leibler approaches the Fisher distance for small  $d$ . In fact, this result is more general, it also holds for multivariate normal distributions when  $P$  approaches  $Q$  in the parameter space [4].

Figure 7 (right) displays the graphics of the mappings  $KL(A || Y)$  (red continuous curve),  $KL(Y || A)$  (blue dashed curve), and the symmetrized  $d_{KL}(A, Y)$  (green dot-dashed curve), compared to the Fisher distance  $d$  (identity) varying in the interval  $[0, 2.3769]$ , with  $Y$  going from  $A$  to  $B$  along the geodesic path of Figure 2.

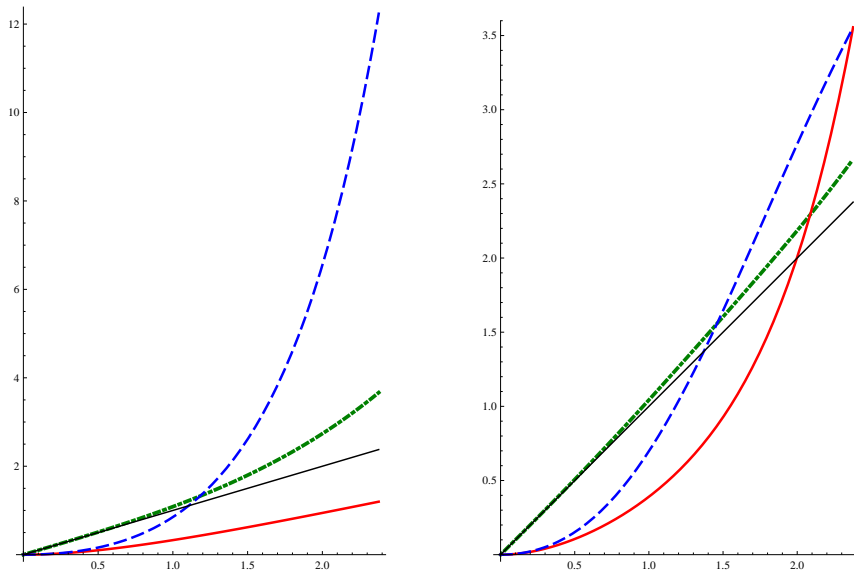


Figure 7: Kullback-Leibler divergences compared to the Fisher distance along the geodesics of Figure 2 connecting the PDF's  $A$  to  $F$  (left) and  $A$  to  $B$  (right).

### 3 Fisher information geometry of multivariate normal distributions

For more general  $p$ -variate PDF's, defined by an  $n$ -dimensional parameter space, the coefficients of the Fisher matrix are given by

$$g_{ij}(\boldsymbol{\beta}) = \int_{\mathbb{R}^p} f(\mathbf{x}, \boldsymbol{\beta}) \frac{\partial \ln f(\mathbf{x}, \boldsymbol{\beta})}{\partial \beta_i} \frac{\partial \ln f(\mathbf{x}, \boldsymbol{\beta})}{\partial \beta_j} d\mathbf{x}.$$

The previous analysis can be extended to independent  $p$ -variate normal distributions:

$$f(\mathbf{x}, \boldsymbol{\mu}, \Sigma) = (2\pi)^{-\frac{p}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp \left( -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right),$$

where

$$\mathbf{x} = (x_1, x_2, \dots, x_p)^T,$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_p)^T \text{ (mean vector) and}$$

$\Sigma$  is the covariance matrix (symmetric positive definite  $p \times p$  matrix).

Note that, for general multivariate normal distributions, the parameter space has dimension  $n = p + p(p+1)/2$ .

#### 3.1 Round Gaussian distributions

If  $\Sigma = \sigma I$  (scalar covariance matrix), the set of all such distributions can be identified with the half  $(p+1)$ -dimensional space,  $\mathbb{H}_F^{p+1}$ , parametrized by  $\boldsymbol{\beta} = (\mu_1, \mu_2, \dots, \mu_p, \sigma)$

and the Fisher information matrix is:

$$\begin{bmatrix} \frac{1}{\sigma^2} & 0 & & & \\ 0 & \frac{1}{\sigma^2} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sigma^2} & 0 \\ & & & 0 & \frac{2}{\sigma^2} \end{bmatrix}.$$

We have again similarity with the matrix of the Poincaré model metric in the  $(p+1)$ -dimensional half space  $\mathbb{H}^{p+1}$ ,

$$\begin{bmatrix} \frac{1}{\sigma^2} & 0 & & & \\ 0 & \frac{1}{\sigma^2} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sigma^2} & 0 \\ & & & 0 & \frac{1}{\sigma^2} \end{bmatrix},$$

and the similarity transformation

$$\Psi : \mathbb{H}_F^{p+1} \longrightarrow \mathbb{H}^{p+1}, \Psi(\mu_1, \mu_2, \dots, \mu_p, \sigma) = (\mu_1/\sqrt{2}, \mu_2/\sqrt{2}, \dots, \mu_p/\sqrt{2}, \sigma).$$

For  $\boldsymbol{\mu}_1 = (\mu_{11}, \mu_{12}, \dots, \mu_{1p})$  e  $\boldsymbol{\mu}_2 = (\mu_{21}, \mu_{22}, \dots, \mu_{2p})$  we have a closed form for the Fisher distance between the respective Gaussian PDF's:

$$d_{F,r}((\boldsymbol{\mu}_1, \sigma_1), (\boldsymbol{\mu}_2, \sigma_2)) = \sqrt{2} d_H \left( \left( \frac{\boldsymbol{\mu}_1}{\sqrt{2}}, \sigma_1 \right), \left( \frac{\boldsymbol{\mu}_2}{\sqrt{2}}, \sigma_2 \right) \right),$$

$$d_{F,r}((\boldsymbol{\mu}_1, \sigma_1), (\boldsymbol{\mu}_2, \sigma_2)) = \sqrt{2} \ln \frac{\left| \left( \frac{\boldsymbol{\mu}_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\boldsymbol{\mu}_2}{\sqrt{2}}, -\sigma_2 \right) \right| + \left| \left( \frac{\boldsymbol{\mu}_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\boldsymbol{\mu}_2}{\sqrt{2}}, \sigma_2 \right) \right|}{\left| \left( \frac{\boldsymbol{\mu}_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\boldsymbol{\mu}_2}{\sqrt{2}}, -\sigma_2 \right) \right| - \left| \left( \frac{\boldsymbol{\mu}_1}{\sqrt{2}}, \sigma_1 \right) - \left( \frac{\boldsymbol{\mu}_2}{\sqrt{2}}, \sigma_2 \right) \right|}$$

where  $|\cdot|$  is the standard Euclidean vector norm and the subindex  $r$  stands for round distributions.

The geodesics in the parameter space  $(\boldsymbol{\mu}, \sigma)$  between two round  $p$ -variate Gaussian distributions are contained in planes orthogonal to the hyperplane  $\sigma = 0$ , and are either a line ( $\mu = \text{constant}$ ) or a half ellipse with eccentricity  $\sqrt{2}$ , centered at this hyperplane.

### 3.2 Diagonal Gaussian distributions

For general  $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_p^2)$  (diagonal covariance matrix),  $\sigma_i > 0, \forall i$ , the set of all independent multivariate normal distributions is parametrized by an intersection of half-spaces in  $\mathbb{R}^{2p}$  ( $\beta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \dots, \mu_p, \sigma_p), \sigma_i > 0$ ) so the Fisher information matrix

is:

$$\begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{\sigma_1^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{2}{\sigma_p^2} \end{bmatrix}.$$

We can show that, in this case, the metric is a product metric on the space  $\mathbb{H}_F^{2,p}$  and therefore we have the following closed form for the Fisher distance between the respective Gaussian PDFs:

$$d_{F,d}((\mu_{11}, \sigma_{11}, \dots, \mu_{1p}, \sigma_{1p}), (\mu_{21}, \sigma_{21}, \dots, \mu_{2p}, \sigma_{2p})) = \sqrt{2} d_{\mathbb{H}^{2p}} \left( \left( \frac{\mu_{11}}{\sqrt{2}}, \sigma_{11}, \dots, \frac{\mu_{1p}}{\sqrt{2}}, \sigma_{1p} \right), \left( \frac{\mu_{21}}{\sqrt{2}}, \sigma_{21}, \dots, \frac{\mu_{2p}}{\sqrt{2}}, \sigma_{2p} \right) \right), \quad (9)$$

that is,

$$\begin{aligned} d_{F,d}((\mu_{11}, \sigma_{11}, \dots, \mu_{1p}, \sigma_{1p}), (\mu_{21}, \sigma_{21}, \dots, \mu_{2p}, \sigma_{2p})) &= \sqrt{\sum_{i=1}^p 2 d_{\mathbb{H}^2} \left( \left( \frac{\mu_{1i}}{\sqrt{2}}, \sigma_{1i} \right), \left( \frac{\mu_{2i}}{\sqrt{2}}, \sigma_{2i} \right) \right)^2} \\ &= \sqrt{\sum_{i=1}^p \left( \ln \frac{\left| \left( \frac{\mu_{1i}}{\sqrt{2}}, \sigma_{1i} \right) - \left( \frac{\mu_{2i}}{\sqrt{2}}, -\sigma_{2i} \right) \right| + \left| \left( \frac{\mu_{1i}}{\sqrt{2}}, \sigma_{1i} \right) - \left( \frac{\mu_{2i}}{\sqrt{2}}, \sigma_{2i} \right) \right|}{\left| \left( \frac{\mu_{1i}}{\sqrt{2}}, \sigma_{1i} \right) - \left( \frac{\mu_{2i}}{\sqrt{2}}, -\sigma_{2i} \right) \right| - \left| \left( \frac{\mu_{1i}}{\sqrt{2}}, \sigma_{1i} \right) - \left( \frac{\mu_{2i}}{\sqrt{2}}, \sigma_{2i} \right) \right|}} \right)^2} \end{aligned} \quad (10)$$

where  $|\cdot|$  is the standard Euclidean vector norm and the subindex  $d$  stands for diagonal distributions.

These matrices induce a metric of constant negative mean curvature (i.e. a hyperbolic metric) which is equal to  $-\frac{1}{2}$  in case 3.1 and to  $-\frac{1}{2(2p-1)}$  in case 3.2. Expressions for the distance and other geometric properties can be deduced using results on product of Riemannian manifolds and relations with Poincaré models for hyperbolic spaces.

### 3.3 General Gaussian distributions

For general  $p$ -variate normal distributions (given by any symmetric positive definite covariance matrices) the analysis is much more complex as pointed out in [2] and far from being fully developed. From the Riemannian geometry viewpoint this is due to the fact that not all the sectional curvatures of their natural parameter space (which is a  $(p + p(p+1)/2)$ -dimensional manifold) provided with the Fisher metric are constant. As an example, for  $p = 2$  we may parametrize the general (elliptical) 2-variate normal distributions by  $\beta = (\sigma_1, \sigma_2, \mu_1, \mu_2, u)$  where  $\sigma_1^2, \sigma_2^2$  are the eigenvalues and  $u$  the turning angle of the eigenvectors of  $\Sigma$ . The level sets of a pair of such PDF's are families of rotated ellipses, see Figure 8.

The Fisher matrix which induces the distance in this parameter space can be deduced as

$$\begin{bmatrix} \frac{2}{\sigma_1^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sigma_2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos^2(u)}{\sigma_1^2} + \frac{\sin^2(u)}{\sigma_2^2} & \frac{\sin(2u)}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) & 0 \\ 0 & 0 & \frac{\sin(2u)}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) & \frac{\cos^2(u)}{\sigma_1^2} + \frac{\sin^2(u)}{\sigma_2^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(\sigma_1^2 - \sigma_2^2)^2}{\sigma_1^2 \sigma_2^2} \end{bmatrix}.$$

We could not derive a general closed form for the associated Fisher distance in this parameter space. Here, like in most multivariate cases, numerical approaches must be used to estimate the Fisher distance. In these approaches, the symmetrized Kullback-Leibler can be used to estimate the Fisher distance between nearby points in the parameter space [4].

A special instance of this bivariate model is given by the set of points with fixed means  $\mu_1, \mu_2$  and turning angle  $u = 0$ . Using the characterization of geodesics as solutions of a second order differential equation [9], we can assert that this two-dimensional submanifold is totally geodesic (i.e. all the geodesics between two of such points are contained in this submanifold). Therefore, the Fisher distance can be calculated as in 3.2:

$$d_F((\sigma_{11}, \sigma_{12}, \mu_1, \mu_2, 0), (\sigma_{21}, \sigma_{22}, \mu_1, \mu_2, 0)) = \sqrt{2} \sqrt{\left( \ln \left( \frac{\sigma_{11}}{\sigma_{12}} \right) \right)^2 + \left( \ln \left( \frac{\sigma_{21}}{\sigma_{22}} \right) \right)^2}.$$

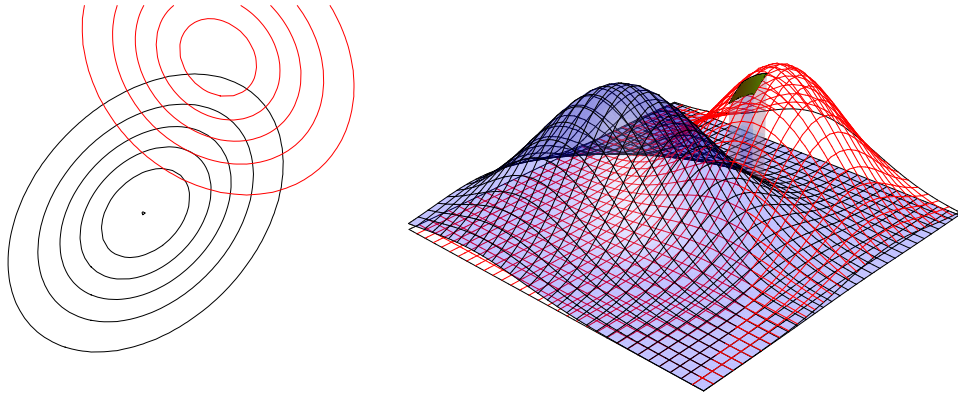


Figure 8: Bivariate normal distributions: level sets (left) and representation in the upper half-space (right).

If we consider the  $(p(p+1)/2)$ -dimensional statistical model of  $p$ -variate normal PDF's with fixed mean  $\boldsymbol{\mu}$  and general covariance matrix  $\Sigma$ , the induced Fisher distance can be deduced as [2]

$$d_F^2((\boldsymbol{\mu}, \Sigma_1), (\boldsymbol{\mu}, \Sigma_2)) = \frac{1}{2} \sum_{j=1}^p (\ln \lambda_j)^2, \quad (11)$$

where  $\lambda_j$  are the eigenvalues of matrix  $(\Sigma_1)^{-1}\Sigma_2$  (i.e.  $\lambda_j$  are the roots of the equation  $\det((\Sigma_1)^{-1}\Sigma_2 - \lambda I) = 0$ ). Note that, for  $p = 1$ , the expression (11) reduces to (7).

Moreover, by restricting (11) to the set of distributions with diagonal covariance matrices, the induced metric is the same as the metric from the case 3.2 restricted to distributions with fixed mean  $\boldsymbol{\mu}$ .

## 4 Final remarks

We have presented a geometrical view of the Fisher distance, focusing on the parameters that describe the normal distributions, to widen the range of possible interpretations for the prospective applications of information geometry, in particular to information theory.

By exploring the two dimensional statistical model of the Gaussian (normal) univariate PDF, we have employed results from the classical hyperbolic geometry to derive closed forms for the Fisher distance in the most commonly used parameters. A relationship with the Kullback-Leibler measure of divergence was presented as well. The multivariate normal PDF's were also analyzed from the geometrical standpoint and closed forms for the Fisher distance were devised in special instances.

## References

- [1] S. Amari and H. Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs, Vol.191, Am. Math. Soc., 2000.
- [2] C. Atkinson and A. F. S. Mitchell, Rao's Distance Measure, *Samkhyā - The Indian Journal of Statistics*, 43:345-365, 1981.
- [3] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York, 1982.
- [4] K. M. Carter, R. Raich and A.O. Hero III, Learning on statistical manifolds for clustering and visualization, *Proceedings of Forty-Fifth Annual Allerton Conference on Communication, Control, and Computing*, 8p., 2007.
- [5] M. H. M. Costa and T.M. Cover, "On the Similarity of the Entropy Power Inequality and the Brunn- Minkowski Inequality," *IEEE Trans. Inform. Theory*, 30(6):837-839, 1984.
- [6] S. I. R. Costa, S. A. Santos and J. E. Strapasson, Fisher information matrix and hyperbolic geometry, *Proc. of IEEE ISOC ITW2005 on Coding and Complexity*, pp.34-36, 2005.
- [7] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley-Interscience Publication, New York, 1999.

- [8] A. Dembo and T. M. Cover, “Information Theoretic Inequalities,” *IEEE Trans. Inform. Theory*, 37(6):1501-1518, 1991
- [9] M. Do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.
- [10] S. Kullback and R. A. Leibler, “On Information and Sufficiency”. *Annals of Mathematical Statistics*, 22(1):79-86, 1951.
- [11] M. Liu, B. C. Vemuri, S. Amari, F. Nielsen, Total Bregman Divergence and its Applications to Shape Retrieval, *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pp.3463-3468, 2010
- [12] M. Lovrić, M. Min-Oo and E. A. Ruh, Multivariate normal distributions parametrized as a Riemannian symmetric space, *Journal of Multivariate Analysis*, 74:36-48, 2000.
- [13] F. Nielsen and V. Garcia, Statistical exponential families: A digest with ash cards, 28p. (v1, 2009) v2, 2011, <http://arxiv.org/abs/0911.4863v2>.
- [14] F. Nielsen and R. Nock, Sided and Symmetrized Bregman Centroids, *IEEE Transactions on Information Theory*, 55(6) 2882-2904, 2009.
- [15] A. M. Peter. and A. Rangarajan, Information Geometry for Landmark Shape Analysis: Unifying Shape Representation and Deformation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 31(2):337-350, 2009.
- [16] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, *Bulletin of the Calcutta Math. Soc.* 37:81-91, 1945.
- [17] L. T. Skovgaard, A Riemannian geometry of the multivariate normal model, *Scand. J. Statist.* 11:211-223, 1984.